An Approximate Analytic Solution to the Restricted Three-Body Problem
Case 105-9

DATE: December 21, 1971

FROM: C. C. H. Tang

ABSTRACT

A mathematical technique to obtain an approximate analytic solution to the trajectory problem of a planetary (or lunar) flyby or impact is presented for the coplanar case. The approximate solution is valid in regions including both the approach phase (transition region) and the encounter phase (dominant planetary or lunar force region). For a spacecraft impacting upon a fictitious planet of point mass, the approximate solution is shown to be nonsingular at the time (or point) of impact. The numerical accuracy of the solution has not been tested, due to lack of availability of a computer program for direct integration of the nonlinear differential equations of the coplanar case.

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MEMORANDUM FOR FILE

I. INTRODUCTION

The trajectory study of a spacecraft moving in solar orbit toward the vicinity of a planet falls in the class of restricted three-body problems. At present, the so-called patched conics method, matched asymptotic expansion method,[1]-[5] or separatrix method[6] can be used to obtain approximate trajectories, whereas final accurate trajectories are always obtained by numerical integration of the equations of motion. The method of numerical integration can yield useful numbers but not the insight that an analytic solution can impart. For planetary (or lunar) flyby or impact problems, there is no known analytic solution which is valid in regions including both the approach phase (transition region) and the encounter phase (dominant planetary or lunar force region). It is the purpose of this paper to show that such an approximate analytic solution in these regions can be obtained by a technique that provides a single continuous solution in these two regions without resorting to matching two solutions.

There is no available method to solve the general non-linear vector equation of motion of a restricted three-body problem, but for planetary flyby or impact problems, we show that the scalar component equations of the vector equation can be "linearized" in the region of interest and can be solved without difficulties of encountering a singularity in the neighborhood of the perturbing body. For the coplanar case in a polar coordinate system, the equations for both the time of flight t and the heliocentric distance r of the spacecraft contain a continued fraction[7] in integral form with θ (the polar angle between r and the reference direction) as the integration variable. The accuracies obtainable in solving the component equations depend on the accuracy of the approximate value (determined by the number of fractions taken into consideration) to be used in evaluating the continued fraction. The linearized differential equation of motion appears in the form

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \varepsilon S(\theta)\,\frac{d}{d\theta}\left(\frac{1}{r}\right) + \left(\frac{1}{r}\right) = F(\theta) ,$$

where ε is a constant much smaller than unity, $S(\theta)$ approaches a very large value or infinity as θ approaches ε or zero, and $F(\theta)$ is the forcing function.

For an arbitrarily prescribed $S(\theta)$, the homogeneous part of the above differential equation cannot be solved exactly by presently known mathematical methods. However, it can be shown that the differential equation can be solved approximately with prescribed accuracy in the region of interest by the method described in references [8] and [9]. The approximate homogeneous solution of the above equation has the form

$$\frac{1}{r} = e^{i\int_{x_i}^{x}\left(\omega + \frac{\varepsilon^2}{2}S\right)dx}\left[A + B\int_{x_i}^{x} e^{-i\int_{x_i}^{x}(2\omega+\varepsilon^2 S)\,dx}\; e^{-\varepsilon\int_{x_i}^{x} S\,dx}\,dx\right],$$

and is accurate to $o(\varepsilon^2)$ in the region of interest where $S(\theta)$ becomes very large or singular as θ approaches zero.

In the following section, we shall show how the imhomogeneous differential equation of motion is obtained, linearized, and solved.

II. MATHEMATICAL FORMULATION

In the sun (mass M) - centered polar coordinate system shown in Fig.1, the vector equation of motion of a spacecraft s (mass m_s) moving, under the influence of the sun, toward the vicinity of a destination planet p (mass m) is

$$\ddot{\underline{r}} = -G(M+m_s)\frac{\underline{r}}{r^3} - Gm\left(\frac{\underline{r}_{ps}}{r_{ps}^3} + \frac{\underline{r}_p}{r_p^3}\right), \tag{1}$$

where G is the universal gravitational constant,
$\underline{r}$ is the vector from the center of mass of the sun to that of the spacecraft,
$\underline{r}_p$ is the vector from the center of mass of the sun to that of the destination planet,
$\underline{r}_{ps}$ is the vector from the center of mass of the planet to that of the spacecraft, and
r, r_p, r_{ps} are respectively the magnitudes of $\underline{r}$, $\underline{r}_p$, $\underline{r}_{ps}$.

Assuming the condition $M \gg m \gg m_s$, we can write Eq.(1) as

$$\ddot{\underline{r}} = -GM\left[\frac{\underline{r}}{r^3} + \mu\left(\frac{\underline{r}_{ps}}{r_{ps}^3} + \frac{\underline{r}_p}{r_p^3}\right)\right]. \tag{2}$$

where $\mu = \frac{m}{M}$. (2a)

Previous studies[1]-[6] have encountered the difficulty that the perturbing term of Eq.(2) apparently becomes very large or singular when r_{ps} approaches zero. In reality, however, the perturbing term may tend to be extremely large but never become singular for the case of a flyby or direct impact, since r_{ps} does not vanish even in the case of direct impact due to the fact that the moon or planets are not point masses. This fact will be exploited in the formulation to be followed.

The vector equation of motion of Eq.(2) can be resolved into components along and perpendicular to the vector $\underline{r}$, for the coplanar case, as:

$$\ddot{r} - r\dot{\theta}^2 = -GM\left[\frac{1}{r^2} + \mu\left(\frac{r}{r_{ps}^3} + Q\cos\beta\right)\right], \tag{3}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -\mu GMQ\sin\beta, \tag{4}$$

where
$$Q = \frac{1}{r_p^2}\left[1 - \left(\frac{r_p}{r_{ps}}\right)^3\right], \tag{5}$$

θ is the angle between the reference direction $\hat{\underline{\gamma}}$ and $\underline{r}$, and

β is the angle between $\underline{r}_p$ and $\underline{r}$ as shown in Fig.1.

Integration of Eq.(4) yields:

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{h_o}{r^2}\left[1 - \frac{\mu GM}{h_o}\int_{t_i}^{t} rQ\sin\beta dt\right], \tag{6}$$

where h_o is the constant of integration or the initial specific angular momentum of the spacecraft with respect to the sun. Inverting and integrating Eq.(6) yields:

$$t = \int_{\theta_i}^{\theta} \frac{r^2}{h_o}\left[1 - \frac{\mu GM}{h_o}\int_{t_i}^{t} rQ\sin\beta dt\right]^{-1} d\theta . \tag{7}$$

Equation (7) is an integral equation in the unknown t to be solved. The nonlinear differential equation of motion in the form of Eq.(3) can be transformed into a more tractable form by letting $r = \frac{1}{y}$ and becomes

$$\frac{d^2y}{d\theta^2} + y = \frac{GM}{h_o^2}\left[1 + \frac{\mu}{y^2}\left(\frac{1}{yr_{ps}^3} + Q\cos\beta + \frac{Q\sin\beta}{y}\frac{dy}{d\theta}\right)\right]\left[1 - \frac{\mu GM}{h_o}\int_{t_o}^{t}\frac{Q\sin\beta}{y}dt\right]^{-2} . \tag{8}$$

Note that Eqs.(7) and (8) are exact and contain the same factor

$$N = \left[1 - \frac{\mu GM}{h_o}\int_{t_i}^{t}\frac{Q\sin\beta}{y}dt\right] . \tag{9}$$

With the aid of Eq.(7), Eq.(9) can be cast into a continued fraction[7] in integral form as

$$N = 1 - \frac{\mu GM}{h_o^2}\int_{\theta_i}^{\theta}\frac{Q\sin\beta d\theta}{y^3\left\{1 - \frac{\mu GM}{h_o^2}\int_{\theta_i}^{\theta}\frac{Q\sin\beta d\theta}{y^3\left[1 - \frac{\mu GM}{h_o^2}\int_{\theta_i}^{\theta}\frac{Q\sin\beta d\theta}{y^3\left(1 - \frac{\mu GM}{h_o}\int_{t_i}^{t}\frac{Q\sin\beta}{y}dt\right)}\right]}\right\}}$$

$$= 1 + \int_{\theta_i}^{\theta}\cfrac{P d\theta}{1 + \int_{\theta_i}^{\theta}\cfrac{P d\theta}{1 + \int_{\theta_i}^{\theta}\cfrac{P d\theta}{1 + \int_{\theta_i}^{\theta}\cfrac{P d\theta}{1 + \int_{t_i}^{t} h_o y^2 P dt}}}} \tag{9a}$$

Neglecting the term $\int_{t_i}^{t} h_o y^2 P dt$, we can simplify Eq.(9a) as

$$N \doteq 1 + \int_{\theta_i}^{\theta} \frac{P d\theta}{1 + \int_{\theta_i}^{\theta} \frac{P d\theta}{1 + \ln\left(1 + \int_{\theta_i}^{\theta} P d\theta\right)}} \tag{9b}$$

where $$P = -\frac{\mu GM}{h_o^2} \frac{Q \sin\beta}{y^3} . \tag{10}$$

The accuracy of N is determined by the number of fractions taken into consideration and will be discussed fully in Appendix I. The accuracies of Eqs.(7) and (8) are in turn determined by the accuracy of N. By using Eqs.(9) and (10), Eqs.(7) and (8) appear respectively in the following neat forms:

$$t = \int_{\theta_i}^{\theta} \frac{d\theta}{h_o N y^2} , \tag{11}$$

$$\frac{d^2 y}{d\theta^2} + \frac{P}{N^2} \frac{dy}{d\theta} + y = F , \tag{12}$$

where $$F = \frac{GM}{h_o^2 N^2}\left[1 + \frac{\mu}{y^2}\left(\frac{1}{y r_{ps}^3} + Q\cos\beta\right)\right] , \tag{12a}$$

is the forcing function.

In an attempt to simplify the problem, we linearize the nonlinear terms in Eqs.(11) and (12) by series expansions about

$\theta=\theta_c$ where the sun, planet, and spacecraft are colinear. In doing so we can express P, N, and F explicitly in terms of θ. In this and the next few paragraphs we shall obtain the series expansions of r, r_p, β, and r_{ps} in terms of θ. As mentioned earlier, the distance r_{ps} between the center of the destination planet and the spacecraft is nonvanishing, even for the case of direct impact with a non-point-mass planet. If we specify the distance between the center of the destination planet and the spacecraft to be s_c at the time when the sun, destination planet, and spacecraft are colinear (i.e., when $\theta=\theta_p=\theta_c$, or $\beta=0$), the zeroth-order approximate solution of the spacecraft orbit is an ellipse

$$r_o = \frac{a_o(1-e_o^2)}{1+e_o \cos(\theta-\omega_o)} = \frac{h_o^2/GM}{1+e_o \cos(\theta-\omega_o)} \tag{13}$$

where a_o is the zeroth-order semi-major axis of the ellipse,

e_o is the zeroth-order eccentricity of the ellipse, and

ω_o is the zeroth-order argument of pericenter of the ellipse.

Note that a_o, e_o, and ω_o of Eq. (13) are obtained by the Lambert Theorem under the initial condition that

$$r(\theta=\theta_i) = r_{\theta_i} , \qquad \text{at } t = t_{\theta_i} , \tag{14}$$

and the terminal condition that

$$r(\theta=\theta_c) = r_p(\theta_p=\theta_c) + s_c , \qquad \text{at } t = t_{\theta_c} , \tag{14a}$$

where s_c can be either positive or negative.

The series expansion of r_o of Eq.(13) about $\theta=\theta_c$ is:

$$r_o = r_o(\theta_c) + r_o'(\theta_c)(\theta-\theta_c) + \frac{1}{2!}r_o''(\theta_c)(\theta-\theta_c)^2 + \frac{1}{3!}r_o'''(\theta_c)(\theta-\theta_c)^3 + \cdots, \tag{15}$$

where prime denotes the derivative with respect to θ.

If we make the substitution $x=\theta-\theta_c$, Eq.(15) becomes

$$r_o = r_o(0)\left(1 + \delta_1 x + \delta_2 x^2 + \delta_3 x^3 + \cdots\right), \tag{16}$$

where

$$\delta_1 = \frac{r_o'(0)}{r_o(0)}, \qquad \delta_2 = \frac{1}{2!}\frac{r_o''(0)}{r_o(0)}, \qquad \delta_3 = \frac{1}{3!}\frac{r_o'''(0)}{r_o(0)}. \tag{17}$$

Note that for $\theta<\theta_c$, x is always negative. For $e_o\leq 0.25$, Eq.(16) is accurate to 10^{-7} of Eq.(13), when $|x|\leq 0.01$ and terms up to x^2 are retained or when $|x|\leq 0.05$ and terms up to x^3 are retained. For $e_o\leq 0.1$, Eq.(16) is accurate to 10^{-8} of Eq.(13) when $|x|\leq 0.01$ and terms up to x^2 are retained, or when $|x|\leq 0.1$ and terms up to x^3 are retained.

The orbit of the destination planet can be represented by an ellipse

$$r_p = \frac{a_p(1-e_p^2)}{1+e_p\cos(\theta_p-\omega_p)} = \frac{a_p(1-e_p^2)}{1+e_p\cos(\theta-\beta-\omega_p)} = \frac{h_p^2/GM}{1+e_p\cos(x-\beta-\omega_p+\theta_c)}, \tag{18}$$

where

a_p is the semi-major axis of the planetary orbit,

e_p is the eccentricity of the planetary orbit,

ω_p is the argument of pericenter of the planetary orbit,

$\theta_p = \theta-\beta$ is the angle between the reference direction $\hat{\gamma}$ and $\underline{r}_p$, and

$h_p = [G(M+m)a_p(1-e_p^2)]^{1/2}$ is the specific angular momentum of the planet with respect to the sun.

The series expansion of r_p of Eq.(18) about x=0 is:

$$r_p = r_p(0)[1 + \Delta_1(x-\beta) + \Delta_2(x-\beta)^2 + \Delta_3(x-\beta)^3 + \cdots] , \tag{19}$$

where

$$\Delta_1 = \frac{r_p'(0)}{r_p(0)} , \qquad \Delta_2 = \frac{1}{2!}\frac{r_p''(0)}{r_p(0)} , \qquad \Delta_3 = \frac{1}{3!}\frac{r_p'''(0)}{r_p(0)} . \tag{20}$$

In order to show r_p of Eq.(19) can be represented explicitly in terms of x, we must also express β in terms of x.

To express β in terms of x, we have to make use of the series expansion of the time of flight of the destination planet and that of the spacecraft; they are, respectively,

$$t_p = t_p(0) + t_p'(0)(x-\beta) + \frac{1}{2!}t_p''(0)(x-\beta)^2 + \frac{1}{3!}t_p'''(0)(x-\beta)^3 + \cdots, \tag{21}$$

$$t_o = t_o(0) + t_o'(0)x + \frac{1}{2!}t_o''(0)x^2 + \frac{1}{3!}t_o'''(0)x^3 + \cdots, \tag{22}$$

where the derivative of time with respect to x can be shown to be (prime denotes the derivative with respect to θ or x)

$$\left.\begin{aligned} & t_p'(0) = r_p^2(0)/h_p, \qquad t_p''(0) = 2\Delta_1\, t_p'(0)\,, \\ & t_p'''(0) = 2(\Delta_1^2+2\Delta_2)t_p'(0), \\ & \text{and} \\ & t_o'(0) = r_o^2(0)/h_o N(0)\,, \qquad t_o''(0) = \left[2\delta_1 - \frac{N'(0)}{N(0)}\right]t_o'(0), \\ & t_o'''(0) = \left[2(\delta_1^2+2\delta_2) - (1+4\delta_1)\frac{N'(0)}{N(0)}\right]t_o'(0) \end{aligned}\right\} \quad (23)$$

where N(0) is defined by Eqs.(9a) and (10) and the zeroth-order approximation of N(0) is unity as shown in Appendix I.

Since $t_p=t_o$ at all times and $t_p(0)=t_o(0)$, we have by equating Eqs.(21) and (22)

$$t_p'(0)(x+\beta) + \frac{1}{2!}t_p''(0)(x+\beta)^2 + \frac{1}{3!}t_p'''(0)(x+\beta)^3 + \cdots =$$

$$t_o'(0)x + \frac{1}{2!}t_o''(0)x^2 + \frac{1}{3!}t_o'''(0)x^3 + \cdots\,. \qquad (24)$$

Neglecting terms containing β^n or x^n with $n \geq 3$ for the sake of simplicity, we obtain the following quadratic equation in β

$$\frac{1}{2}[t_p''(0) + t_p'''(0)x + \frac{1}{2}t_p''''(0)x^2]\beta^2 - [t_p'(0) + t_p''(0)x + \frac{1}{2}t_p'''(0)x^2]\beta$$

$$+ [t_p'(0) - t_o'(0)]x + \frac{1}{2}[t_p''(0) - t_o''(0)]x^2 = 0\,. \qquad (25)$$

The solution of Eq.(25) is approximately

$$\beta = -(k_1x + k_2x^2)\,, \qquad (26)$$

where

$$k_1 = \frac{t_o'(0)}{t_p'(0)} - 1 = \frac{h_p \ell^2}{h_o N(0)} - 1 \ , \tag{27}$$

and

$$k_2 = \frac{1}{2t_p'(0)}\left[t_o''(0) - t_p''(0)\left(\frac{t_o'(0)}{t_p'(0)}\right)^2\right] = (1+k_1)\left[\delta_1 - \Delta_1(1+k_1) - \frac{N'(0)}{2N(0)}\right] . \tag{27a}$$

with

$$\ell = \frac{r_o(0)}{r_p(0)} = 1 + \frac{s_c}{r_p(0)} . \tag{28}$$

Note that if β^3 and x^3 terms were retained, we would have to solve a cubic equation, and the solution would be more accurate for larger values of x.

Equation (19) then becomes, by retaining terms up to x^2,

$$r_p = r_p(0)(1 + D_1 x + D_2 x^2) \ , \tag{29}$$

where $D_1 = \Delta_1(1+k_1)$, and $D_2 = \Delta_1 k_2 + \Delta_2(1+k_1)^2$. (30)

The distance from the center of the planet to the spacecraft is

$$r_{ps} = [r_p^2 + r^2 - 2rr_p \cos\beta]^{1/2} \tag{31}$$

In terms of Eqs.(16), (26) and (29), Eq. (31) can be approximated, by retaining terms up to x^2, as

$$r_{ps} = r_p(x)(a+bx+cx^2)^{\frac{1}{2}} \ , \tag{32}$$

$$\left.\begin{aligned} &\text{where} \quad a = (\ell-1)^2 \ , \qquad b = 2\ell(\ell-1)(\delta_1 - D_1) \ , \quad \text{and} \\ &c = [\ell(\ell_1^2+2\delta_2+3D_1^2-2D_2-4\delta_1 D_1) - 2(\delta_2-D_2+D^2-\delta_1 D_1 - \tfrac{1}{2}k_1^2)] \end{aligned}\right\} \tag{33}$$

Note that r_{ps} does not vanish and is equal to $r_o(0)-r_p(0)$ as x approaches zero. Note also that a=b=0 when $\ell=1$, i.e., when $r_o(0)=r_p(0)$ or $s_c=0$ for a fictitious point-mass planet.

With the expansions introduced in Eqs.(16), (26), (29), and (32), Eq.(10) becomes

$$P = -\frac{\mu GM}{h_o^2}\frac{Q\sin\beta}{y^3} = \varepsilon U(x) \quad , \tag{34}$$

where

$$\varepsilon = \frac{\mu GM}{h_o^2} k_1 r_p(0)\ell^3 \qquad \text{and} \qquad U(x) = x(1+fx)\left[1 - \frac{1}{(a+bx+cx^2)^{3/2}}\right]. \tag{35}$$

Similarly, Eqs.(11) and (12) become

$$t = \frac{r_p^2(0)\ell^2}{h_o}\int_{x_i}^{x}\frac{(1+\delta_1 x+\delta_2 x^2)^2}{N(x)}dx \quad , \tag{36}$$

and

$$\frac{d^2y}{dx^2} + \varepsilon S(x)\frac{dy}{dx} + \omega^2 y = F(x) = \frac{GM}{h_o^2 N^2(x)}\left\{1+\mu\left[\ell^3\frac{(1+b_1x+c_1x^2)}{(a+bx+cx^2)^{3/2}}\right.\right.$$

$$\left.\left. + \ell^2(1+b_2x+c_2x^2)\left(1 - \frac{1}{(a+bx+cx^2)^{3/2}}\right)\right]\right\} \quad , \tag{37}$$

where

$$S(x) = U(x)/N^2(x) \quad , \tag{37a}$$

Note that U, and therefore S, vanish as x approaches zero since a is nonvanishing for $|s_c|>0$.

and

$$\left.\begin{array}{ll}\omega = 1\ , & f = 3\delta_1 - 2D_1 + \dfrac{k_2}{k_1}\ , \\[2ex] b_1 = 3(\delta_1 - D_1)\ , & c_1 = 3(\delta_2 - D_2 + \delta_1^2 + 2D_1^2 - 3\delta_1 D_1)\ , \\[2ex] b_2 = 2(\delta_1 - D_1)\ , & c_2 = 2(\delta_2 - D_2) + \delta_1^2 + 3D_1^2 - 4\delta_1 D_1 - \frac{1}{2}k_1^2\ . \end{array}\right\} \quad (38)$$

It is important to note that as x approaches zero, $\varepsilon S(x)$ and $F(x)$ do not display the troublesome singular behavior that was encountered by previous investigators[1]-[6] of restricted three-body problems. In fact, $S(x)$ approaches zero as x does except in the case $\ell=1$ $\left(s_c=0\right)$, when $S(x)$ varies as $-\ c^{-3/2}\ x^{-2}$ as x approaches zero. This case will be discussed later.

It turns out that the following integral can be integrated

$$\eta(x) = \int_{x_i}^{x} P dx = \varepsilon \int_{x_i}^{x} U(x) dx = \varepsilon\left[\left(\frac{1}{2} + \frac{1}{3}\,fx\right)x^2\right. \quad (39)$$

$$\left. - \frac{2a(bf-2c) + [f(b^2-g) - 2bc]x}{cg\left(a+bx+cx^2\right)^{1/2}} - \frac{b}{c^{3/2}}\sinh^{-1}\frac{(b+2cx)}{\sqrt{g}}\right]_{x_i}^{x}\ ,$$

where $$g = 4ac - b^2 > 0. \quad (39a)$$

For realistic cases where a and b are always non-vanishing, we show in Appendix I that $\eta(x)$ is always positive for negative values of x, and less than unity. With the aid of Eq. (39), the evaluation of $N(x)$ of Eq. (9b) can be simplified somewhat.

Although Eq.(37) is linear, there is no available analytic method to solve the homogeneous part of the equation exactly due to the presence of the variable damping coefficient $\varepsilon S(x)$. One possible approximate method is to consider the damping term $\varepsilon S(x)\frac{dy}{dx}$ as a perturbation term and apply Poincare's first-order small-parameter expansion. This approximation method, however, would not be accurate enough for our purpose since $\varepsilon S(x)$ for the case of a close flyby could be a few orders of magnitude larger than $\omega^2=1$ in the region where x approaches zero. Figure A2 in Appendix I should represent the order-of-magnitude variation of $\varepsilon S(x)$ very closely since $\varepsilon S(x)= P(x)/N^2(x)$, where $N^2(x)$ ranges between unity and about two, as shown in Appendix I. To gain some physical insight we also plot both $\varepsilon S(x)\frac{dy}{dx}$ and $F(x)$ in Fig.2 for different values of s_c. Note the steep variations and change of sign of $F(x)$ in the neighborhood of the sphere of influence for cases $s_c=10^{-4}$ and 0.25×10^{-4}. Also note that there is always a region where $\varepsilon S(x)\frac{dy}{dx}$ and $F(x)$ are of the same order of magnitude.

An approximate method to solve a differential equation similar to the homogeneous part of Eq.(37) has recently been developed[8]-[9]. The approximate homogeneous solution thus obtained is accurate to $o(\varepsilon^2)$ and has the form[9]

$$y_h = Ay_1 + By_2 , \tag{40}$$

$$y_1 = e^{i\int_{x_i}^{x}(\omega + \frac{\varepsilon^2}{2}S)dx} , \tag{40a}$$

$$y_2 = y_1\int_{x_i}^{x} [y_1\xi(x)]^{-1}\, dx = y_1 I(x) , \tag{40b}$$

with

$$I(x) = \int_{x_i}^{x} [y_1 \xi(x)]^{-1} dx \ , \qquad \xi(x) = y_1 e^{\varepsilon \int_{x_i}^{x} S dx} \ , \tag{40c}$$

and

$$A = y_h(x_i) \ ,$$

$$B = y_h'(x_i) - i\left[\omega + \frac{\varepsilon^2}{2} S(x_i)\right] y_h(x_i) \ . \tag{40d}$$

where $y_h(x_i)$ and $y_h'(x_i)$ are the zeroth-order trial initial conditions obtained from Eqs.(13) and (14). The approximate solution, Eq.(40), is accurate to $o(\varepsilon^2)$ as shown in reference [9]. The relevant important results of reference [9] are summarized in Appendix II.

The particular solution of Eq.(37) has the following form:

$$y_p = \frac{1}{K}\int_{x_i}^{x} [y_1(\alpha)\ y_2(x) - y_2(\alpha)\ y_1(x)] F(\alpha)\ e^{\varepsilon \int_{\alpha_i}^{\alpha} S(\alpha) d\alpha} d\alpha, \tag{41}$$

where K is a constant obtained from Abel's formula for the Wronskian. In this case,

$$K = e^{\varepsilon \int_{x_i}^{x} S(x) dx} [y_1(x)\ y_2'(x) - y_2(x)\ y_1'(x)] = 1. \tag{41a}$$

The approximate general solution of Eq.(37) therefore is

$$y = y_h + y_p = \left\{\left[A - \int_{x_i}^{x} F(x)\xi(x)I(x)dx\right] + \left[B + \int_{x_i}^{x} F(x)\xi(x)dx\right] I(x)\right\} y_1(x)$$

$$= y_1(x)\left[A + \int_{x_i}^{x} \frac{B + \int_{x_i}^{x} F(x)\xi(x)dx}{\xi(x)y_1(x)} dx\right] . \qquad (42)$$

The derivatives of the solution can be obtained neatly as follows

$$y' = i\left[\omega + \frac{\varepsilon^2}{2} S(x)\right] y + \frac{1}{\xi(x)}\left[B + \int_{x_i}^{x} F(x)\xi(x)dx\right] , \qquad (43)$$

$$y'' = \left\{-\left[\omega + \frac{\varepsilon^2}{2} S(x)\right]^2 + i \frac{\varepsilon^2}{2} S'(x)\right\} y$$

$$- \frac{\varepsilon S(x)}{\xi(x)}\left[B + \int_{x_i}^{x} F(x)\xi(x)dx\right] + F(x) . \qquad (44)$$

Equation (44) can be expressed in terms of y' of Eq.(43) as:

$$y'' = \left\{-\left[\omega + \frac{\varepsilon^2}{2} S(x)\right]^2 + i\left[\varepsilon S(x)\left(\omega + \frac{\varepsilon^2}{2} S(x)\right) + \frac{\varepsilon^2}{2} S'(x)\right]\right\} y(x)$$

$$- \varepsilon S(x)\ y'(x) + F(x) . \qquad (45)$$

Comparison of Eq.(45) with the original differential equation (37) indicates that Eq.(45) is in error only in the coefficient of y by a very small real part $\frac{\varepsilon^2}{2} S(x)$ and an imaginary part $\left[\varepsilon S(x)\left(\omega + \frac{\varepsilon^2}{2} S(x)\right) + \frac{\varepsilon^2}{2} S'(x)\right]$. These are in agreement with the results obtained in reference [9]. At x=0, S(0)=0, $S'(0) = \frac{U'(0)}{N^2(0)} = -\frac{1}{a^{3/2}N^2(0)}$ and Eq.(45) becomes

$$y''(0) = -\left[\omega^2 + i\,\frac{\varepsilon^2}{2a^{3/2}N^2(0)}\right]y(0) + F(0) \doteq -\omega^2 y(0) + F(0) \ . \quad (46)$$

The approximate second derivative of Eq.(45) is therefore identical to the original differential Eq.(37) at x=0.

The spacecraft distance from the sun and its derivatives with respect to x can thus be obtained from Eqs.(42) through (44) as follows:

$$r = \frac{1}{y}\ , \qquad r' = -\frac{y'}{y^2}\ , \qquad r'' = -\frac{1}{y}\left[\frac{y''}{y} - 2\left(\frac{y'}{y}\right)^2\right] \ . \quad (47)$$

It is evident, however, that due to the presence of the perturbing destination planet, y(0) thus obtained from Eq.(42) will not meet the required terminal condition specified in Eq.(14a) because the initial conditions used in Eq.(40d) are obtained by ignoring the presence of the destination planet. To meet the terminal condition of this targeting problem, or two-point boundary value problem, the initial conditions used in Eq.(40d) must be perturbed. Methods, such as the steepest descent method, to perturb the initial conditions in the most efficient way to meet the terminal conditions will not be discussed in this paper.

Equation (42) is the formal approximate solution of Eq.(37) and the accuracy of the solution can be examined only by comparing this solution with that obtained by numerically integrating the original nonlinear second order differential equation (2) or other equivalent. Lacking such a program, we cannot carry out the numerical comparison.

Before investigating the case of a fictitious planet of point mass, we reiterate the general applicability of Eqs.(36) and (37) to the case of a spacecraft colliding with the actual destination planet. It is quite clear that, since the planet has a finite size, Eqs.(36) and (37) can be used without any difficulty as they are. In this case s_c might be slightly smaller than the radius of the destination planet.

III. FICTITIOUS PLANET OF POINT MASS

Here we shall investigate the special case of a spacecraft colliding with a fictitious planet of point mass, since this corresponds directly to the case assumed by previous investigators[1]-[6], who have had mathematical difficulties in obtaining solutions in the vicinity of an apparent essential singularity of their differential equations. We shall show that when the singularity of such an equation is approached properly, we can always obtain a non-singular solution even at the singular point. We shall first derive the differential equation for this case from Eqs.(36) and (37). At the time of collision, namely, as $x \to 0$, the distance s_c between the spacecraft and the fictitious point-mass plant approaches zero and from Eq.(28) we have $\ell=1$. In turn we have from Eq.(33) the coefficients $a=b=0$ and $c=(\delta_1-D_1)^2+k_1^2$, and

Eqs.(36) and (37) can be simplified, respectively, as

$$t = \frac{r_p^2(0)}{h_o}\int_{x_i}^{x} \frac{\left(1+\delta_1 x+\delta_2 x^2\right)^2}{N(x)}\, dx \ , \tag{48}$$

$$\frac{d^2y}{dx^2} + \varepsilon\, \frac{x(1+fx)}{N^2(x)}\left(1 - \frac{1}{c^{3/2}x^3}\right)\frac{dy}{dx} + \omega^2 y = \tag{49}$$

$$\frac{GM}{h_o^2 N^2(x)}\left\{1+\mu\left[(1 + b_2x + c_2x^2) + \frac{(\delta_1-D_1) + \left(\delta_2-D_2 + \delta_1^2-\delta_1 D_1 + \frac{k_1^2}{2}\right)x}{c^{3/2}x^2}\right]\right\}.$$

To study the characteristics of Eqs.(48) and (49) as $x \to 0$, we simplify these equations as

$$t = \frac{r_p^2(0)}{h_o}\int_{x_i}^{x} \frac{dx}{N_I(x)} \ , \tag{50}$$

$$\frac{d^2y}{dx^2} - \varepsilon\, \frac{U_I(x)}{N_I^2(x)}\, \frac{dy}{dx} + \omega^2 y = \frac{GM}{h_o^2 N_I^2(x)}\left[1 + \mu\left(1 + \frac{\delta_1-D_1}{c^{3/2}x^2}\right)\right] = F_I(x) \ , \tag{51}$$

$$\left.\begin{aligned} \text{where} \quad & U_I(x) \to \frac{1}{c^{3/2}x^2} \ , && \text{as } x \to 0 \ , \\ & N_I(x) \to 1 + o(0) \ , && \text{as } x \to 0 \ . \end{aligned}\right\} \tag{52}$$

Although both the coefficients of $\frac{dy}{dx}$ and $F_I(x)$ of Eq.(51) are singular as $x \to 0$, the solution in the form of Eq.(42) is nonsingular due to the fact that the integrand of Eq. (42),

$$\frac{B + \int F_I(x)\,\xi(x)\,dx}{\xi(x)y_1(x)} \xrightarrow[x\to 0]{} \frac{F_I(x)\,\xi(x)}{\xi'(x)y_1(x) + \xi(x)y_1'(x)}$$

$$= \frac{F_I(x)\,\xi(x)}{\xi(x)y_1(x)\,[i(\xi\omega+\omega+\varepsilon^2 s/2)+\varepsilon s]} \doteq \frac{F_I(x)N_I^2(x)}{\varepsilon U_I(x)y_1(x)}, \tag{53}$$

remains finite as x 0. Since both $U_I(x)$ and $F_I(x)$ vary as x^{-2} as x 0, their ratio in Eq. (53) is nonsingular. Accordingly, there is no singularity in the solution even for the case of a direct impact of the spacecraft with a fictitious planet of point mass.

IV. SUMMARY

An approximate analytic solution to the trajectory problem of a spacecraft moving, under the influence of the sun (or earth), to the vicinity of a planet (or moon) has been presented for the coplanar case. The solution is valid in both the approach phase (transition region) and the encounter phase (dominant planetary or lunar force region). The solution is shown to be nonsingular at the time (or point) of impact, even for the case of fictitious planet of point mass. The numerical accuracy of the solution has not yet been tested, due to lack of availability of a computer program for direct integration of the nonlinear differential equations of the coplanar case.

V. ACKNOWLEDGMENT

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Attachments
- Appendices I and II
- References
- Figures 1 and 2

APPENDIX I

A CONTINUED FRACTION IN INTEGRAL FORM

The purpose of this appendix is to show that the time of flight expressed in the following integral equation

$$t = \int_{\theta_i}^{\theta} \frac{r^2(\theta)}{h_o} \left[1 - \frac{\mu GM}{h_o} \int_{t_i}^{t} r(\theta) Q(\theta) \sin\beta(\theta) dt \right]^{-1} d\theta \qquad (7)$$

can be solved approximately by the method of continued fraction and the accuracy of the solution t depends on the number of fractions taken into consideration in evaluating the integral.

Differentiating Eq. (7) with respect to θ, we have

$$dt = \frac{r^2(\theta)}{h_o} \left[1 - \frac{\mu GM}{h_o} \int_{t_i}^{t} r(\theta) Q(\theta) \sin\beta(\theta) dt \right]^{-1} .$$

Substitution of the above equation into Eq. (7) repeatedly for four times yields

$$t = \int_{\theta_i}^{\theta} \frac{r^2(\theta)}{h_o N(\theta)} d\theta \ , \qquad (11)$$

where N is shown in Eq. (9a).

Further repeated substitutions will introduce more fractions and more accuracy in N, but we choose only up to four substitutions here in order to limit the complexity in evaluating N.

Equation (9a) is exact, but its approximate evaluation cannot be carried out unless the term $\int_{t_i}^{t} h_o y^2 P dt$ can be neglected. From Eq.(7) we know that $\frac{\mu GM}{h_o}\int_{t_i}^{t} r\theta \sin\beta \, dt = \int_{t_i}^{t} h_o y^2 P dt$ is a perturbation term and therefore in general its absolute value should be less than unity which is the unperturbed term. Unless $\int_{t_i}^{t} h_o y^2 P dt$ is much less than unity the accuracy of N depends on the number of fractions used, as demonstrated in the following case

$$N_0 = 1$$

$$N_1 = 1 + \int_{x_i}^{x} P dx = 1.116{,}181{,}855$$

$$N_2 = 1 + \ln\left(1 + \int_{x_i}^{x} P dx\right) = 1.109{,}913{,}806$$

$$N_3 = 1 + \int_{x_i}^{x} \frac{P dx}{1 + \ln\left(1 + \int_{x_i}^{x} P dx\right)} = 1.110{,}124{,}055$$

$$N_4 = 1 + \int_{x_i}^{x} \frac{P dx}{1 + \int_{x_i}^{x} \frac{P dx}{1 + \ln\left(1 + \int_{x_i}^{x} P dx\right)}} = 1.110{,}118{,}675$$

$$N_5 = \ldots$$

TABLE AI

N AND t ACCURACY COMPARISON

Common Parameters	$x_i = -0.01$, $x = 0$, $\mu = 0.32369\times10^{-6}$ (Mars), $GM = 0.295913\times10^{-3}$ (a.u.)^3/day^2 $a_p = 1.5914690$ a.u., $e_p = 0$, R = radius		
s_c (a.u.)	10^{-3} (44R)	10^{-4} (4.4R)	0.25×10^{-4} (1.1R)
a_o (a.u.)	1.334,011,000	1.332,574,000	1.332,454,000
e_o	0.238,284,200	0.238,142,400	0.238,131,500
ω_o (deg.)	151.730,843	151.972,132	151.992,166
N_0	1.	1.	1.
N_2	1.010,377,590	1.109,913,806	1.383,419,071
N_4	1.010,377,776	1.110,118,675	1.390,855,751
⋮			
N_3	1.010,377,778	1.110,124,055	1.391,487,836
N_1	1.010,431,625	1.116,181,855	1.467,292,800
t_0 (day)	1.312,353,898	1.311,543,912	1.311,478,094
t_2	1.306,661,757	1.303,101,740	1.303,121,932
t_4			
⋮			
t_3	1.306,661,635	1.303,094,916	1.303,075,164
t_1	1.306,639,661	1.302,857,424	1.302,549,041

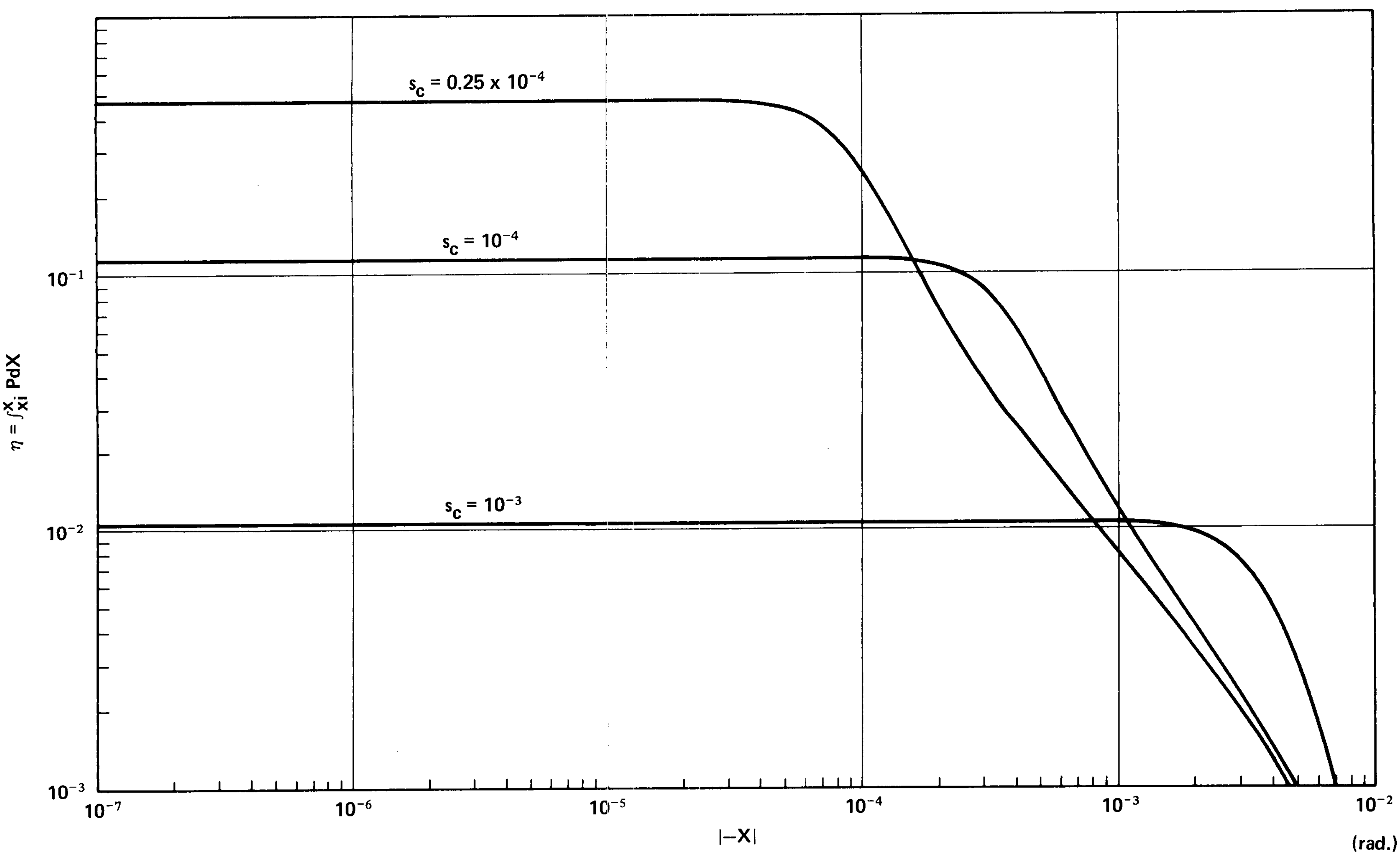

FIGURE A1 - VARIATION OF $\int_{xi}^{x} PdX$ AGAINST $|-X|$

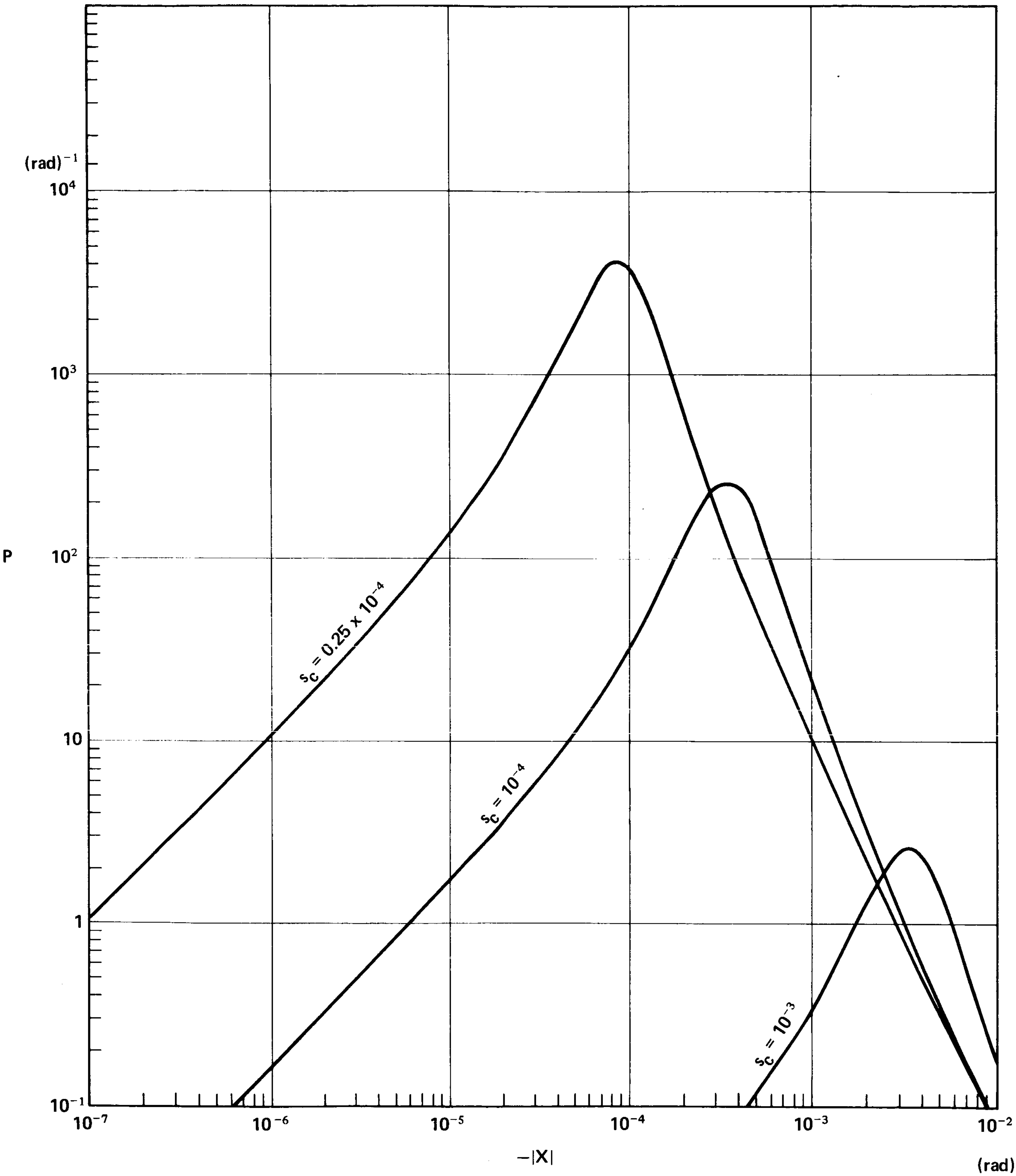

FIGURE A2 - VARIATION OF P AGAINST $|-X|$

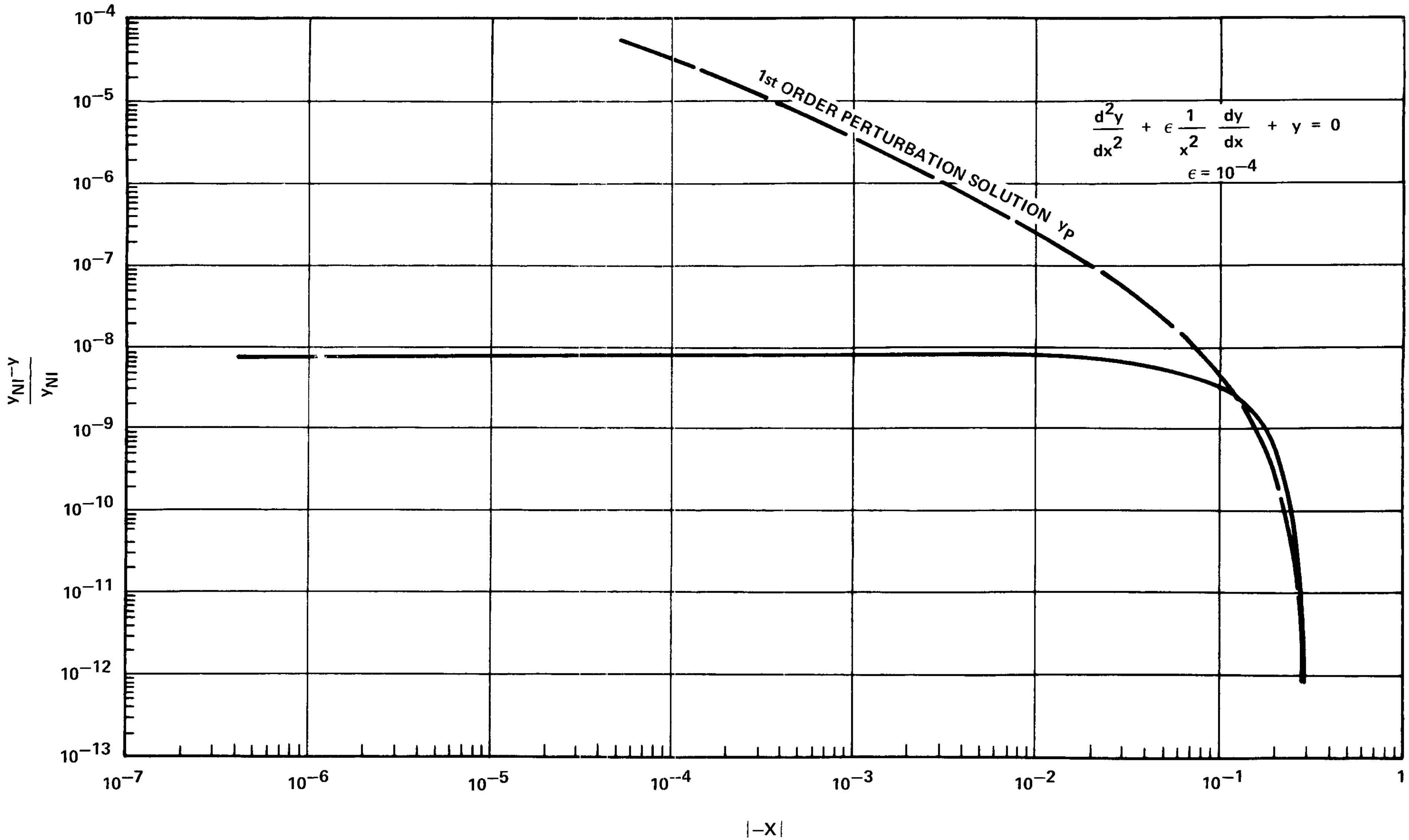

FIGURE A3 - COMPARISON OF SOLUTION ACCURACIES

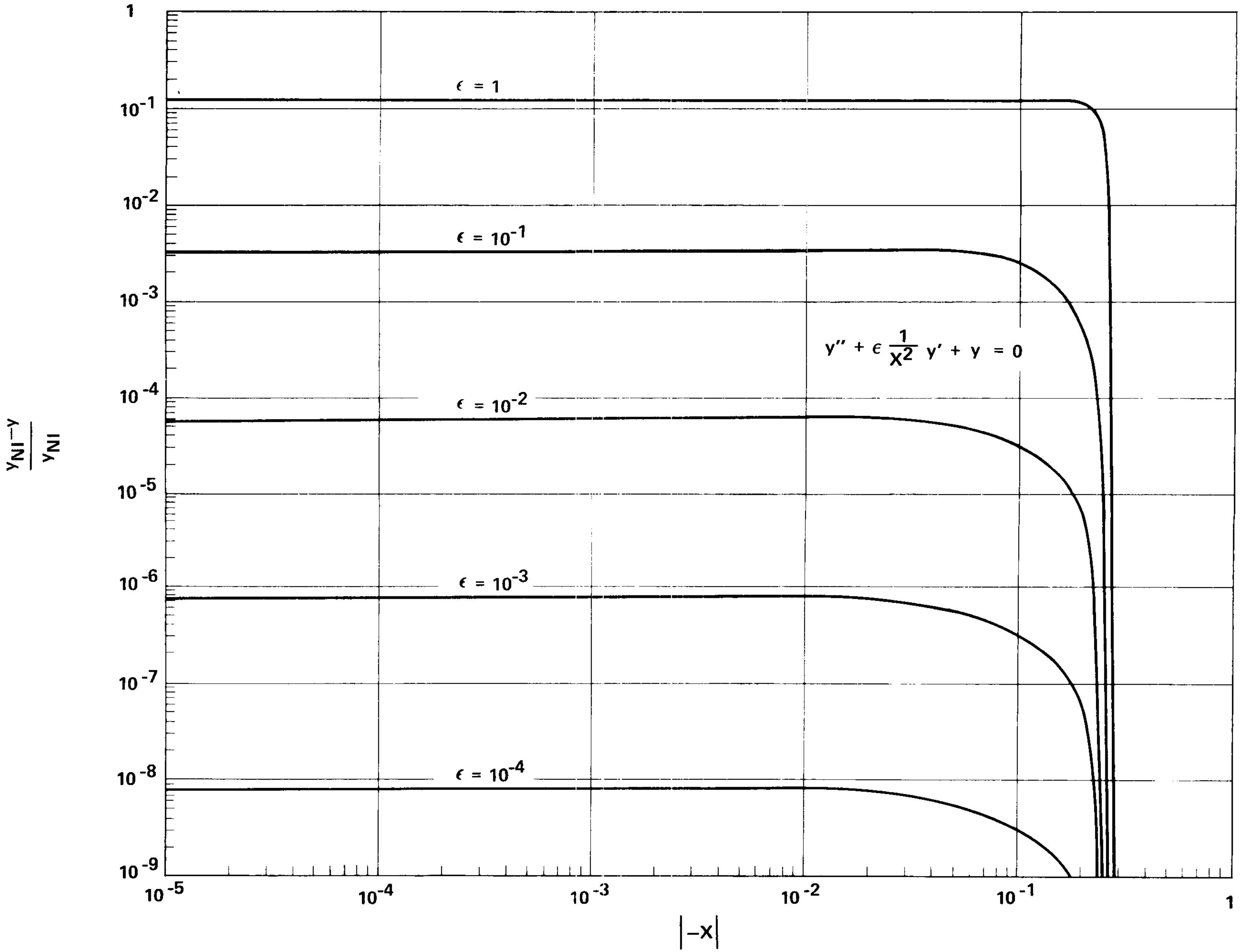

FIGURE A4 - EFFECT OF ϵ ON THE ACCURACY OF THE APPROXIMATE SOLUTION

APPENDIX II

DIFFERENTIAL EQUATIONS WITH PERTURBING SINGULAR DAMPING

Many physical problems are characterized by the presence of a perturbing force which can be either constant or varying. The exact solution of a simple linear oscillator with constant damping is well known, but that with arbitrarily varying damping is unobtainable without resorting to numerical integration. For example, the following differential equation for a linear oscillator with a perturbing singular damping term cannot be solved exactly by presently known methods and functions

$$\frac{d^2y}{dx^2} + \varepsilon\frac{1}{x^2}\frac{dy}{dx} + \omega^2 y = 0 \tag{A1}$$

where ε is a positive real parameter much less than unity and ω^2 a positive real constant of $o(1)$. It can be shown that for $|x| > \sqrt{\varepsilon}$ Poincaré's perturbation method of small parameter expansion will yield a first order homogeneous solution accurate to $o\left(\varepsilon^2 \frac{\ln x}{x}\right)$ which is singular as $x \to 0$. For $0 < |x| < \sqrt{\varepsilon}$ higher order perturbation solution cannot improve the accuracy of the solution in the neighborhood of the singular point at $x=0$, because of the apparent singular nature of the perturbing term in Eq. (A1).

With the aid of a variable transformation, an approximate solution of Eq. (A1) can be shown[9] to have the form

$$y = e^{i\int_{x_i}^{x}\left(\omega + \frac{\varepsilon^2}{2}\frac{1}{x^2}\right)dx}\left[C_1 + C_2\int_{x_i}^{x} e^{-\int_{x_i}^{x}\frac{\varepsilon}{x^2}dx} e^{-i\int_{x_i}^{x}\left(2\omega + \frac{\varepsilon^2}{x^2}\right)dx} dx\right] \tag{A2}$$

$$\left.\begin{aligned} &\text{where} \quad C_1 = y(t_i) \\ &\text{and} \quad C_2 = y'(t_i) + i\left[\omega + \frac{\varepsilon^2}{2}\frac{1}{x_i^2}\right]y(t_i) \end{aligned}\right\} \tag{A3}$$

It is important to note that the solution in the form of Eq.(A2) is nonsingular as $x \to 0^-$ from negative values.

In this appendix we present results from Reference 9 showing that Eq.(A2) is at least accurate to $0(\varepsilon^2)$, as $x \to 0^-$ from negative values, by comparing it with the "exact" solution obtained by numerically integrating Eq.(A1). Fig.A3 shows, for the case $\varepsilon = 10^{-4}$, the comparison of relative difference between the approximate solution y and the numerically integrated solution y_{NI}. The perturbation solution is also shown in Fig.A3 for comparison. The perturbation solution is computed only up to $x = -10^{-4}$, since the first order perturbation solution is no longer valid for $|x| < 10^{-4}$ and its error increases as $o\left(\varepsilon^2 \frac{\ln x}{x}\right)$. It is seen by comparison that the approximate solution is orders of magnitude more accurate than the first order perturbation solution in the region near the apparent singular point. Fig.A4 shows that the variation of accuracy of the approximate solution is at least of the order of ε^2. Table AII shows the corresponding numerical values used in plotting Fig.A4.

TABLE A II NUMERICAL COMPARISON OF SOLUTIONS

X		$\epsilon = 10^{-4}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-2}$	$\epsilon = 10^{-1}$	$\epsilon = 1$
-0.3	$y=$	0.863,209,367	0.863,209,367	0.863,209,367	0.863,209,367	0.863,209,367
-0.2		0.808,500,274	0.808,535,032	0.808,875,614	0.811,604,511	0.726,263,869
-10^{-1}		0.745,730,229	0.745,954,706	0.748,134,341	0.764,482,024	0.708,624,475
-10^{-2}		0.682,963,377	0.684,381,156	0.696,577,684	0.747,712,320	0.705,580,025
-10^{-3}		0.676,522,141	0.679,156,911	0.695,413,136	0.747,712,268	"
-10^{-4}		0.676,000,122	0.679,046,496	0.695,413,133	"	"
-10^{-5}		0.675,989,161	"	"	"	"
-10^{-6}		0.675,989,161	"	"	"	"
-0.3	$y_{NI}=$	0.863,209,367	0.863,209,367	0.863,209,367	0.863,209,367	0.863,209,367
-0.2		0.808,500,274	0.808,535,092	0.808,881,567	0.812,176,793	0.833,280,389
-10^{-1}		0.745,730,232	0.745,955,006	0.748,163,625	0.766,848,989	0.827,964,623
-10^{-2}		0.682,963,385	0.684,381,924	0.696,644,755	0.750,960,965	0.827,626,744
-10^{-3}		0.676,522,149	0.679,157,691	0.695,478,970	0.750,957,898	0.827,626,464
-10^{-4}		0.676,000,130	0.679,047,258	0.695,478,939	0.750,957,896	"
-10^{-5}		0.675,989,168	"	"	"	"
-10^{-6}		0.675,989,168	"	"	"	"
-0.3	$y_{NI}-y=$	0	0	0	0	0
-0.2		0.059×10^{-8}	0.060×10^{-6}	0.059×10^{-4}	0.051×10^{-2}	0.107
-10^{-1}		0.301×10^{-8}	0.300×10^{-6}	0.293×10^{-4}	0.237×10^{-2}	0.119
-10^{-2}		0.778×10^{-8}	0.768×10^{-6}	0.671×10^{-4}	0.325×10^{-2}	"
-10^{-3}		0.797×10^{-8}	0.780×10^{-6}	0.658×10^{-4}	"	"
-10^{-4}		0.781×10^{-8}	0.762×10^{-6}	"	"	"
-10^{-5}		0.763×10^{-8}	0.761×10^{-6}	"	"	"
-10^{-6}		"	"	"	"	"

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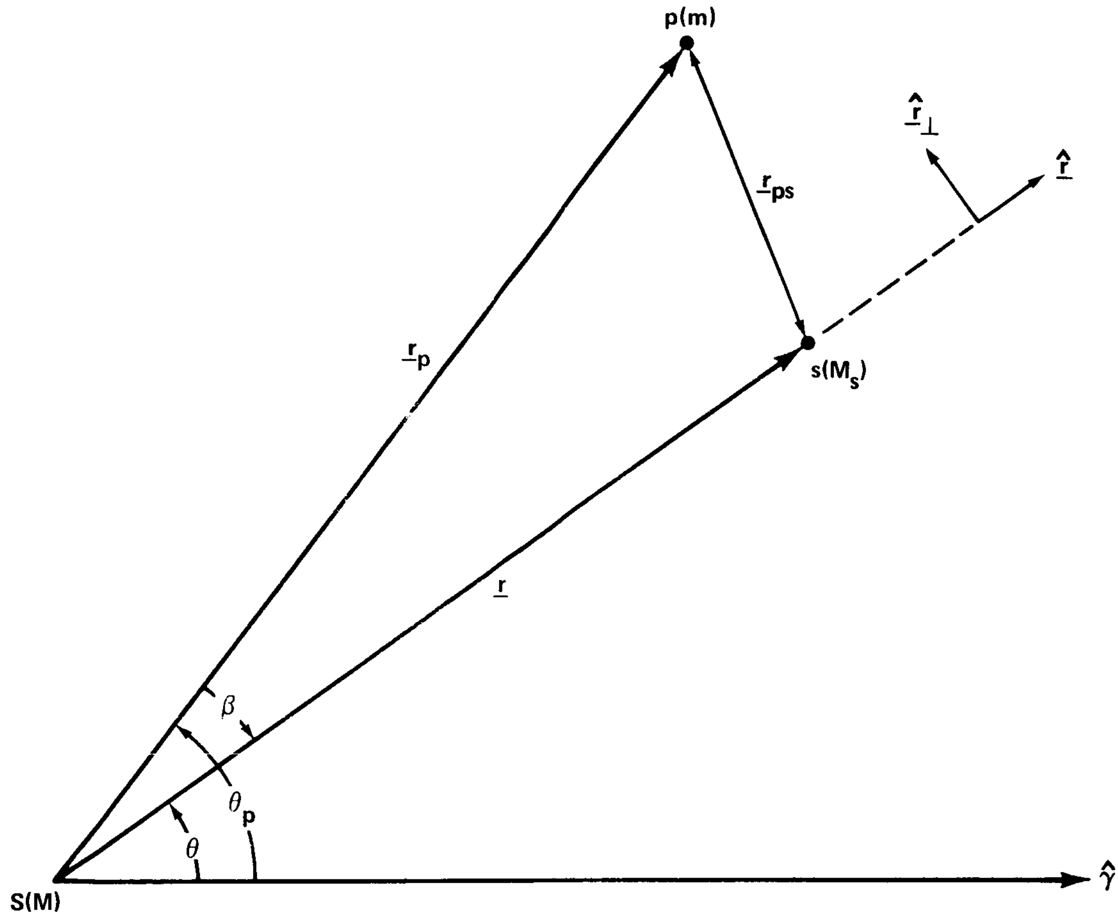

FIGURE 1 - THREE-BODY COPLANAR CONFIGURATION IN POLAR COORDINATE SYSTEM

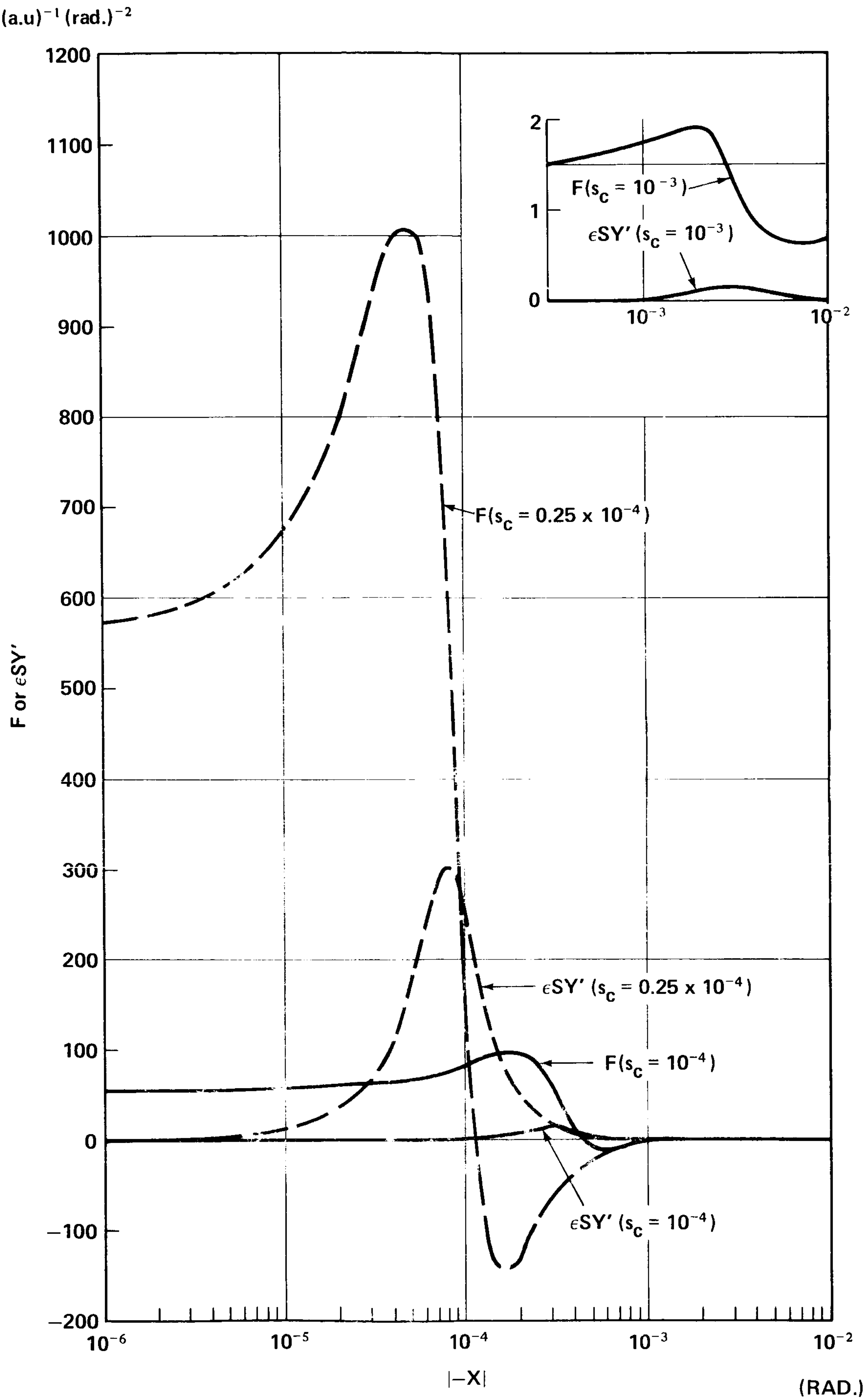

FIGURE 2 - VARIATION OF F AND ϵSY' AGAINST |−X|

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